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THREE-VORTEX MOTION WITH ZERO TOTAL CIRCULATION

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Motion of three point vortices with zero total circulation in an ideal liquid is studied. Vortex trajectories are studied numerically over a wide range of their circulations at various initial positions. Asymptotics of vortex motions are calculated for configurations close to singular and collinear ones.

Key words: *point vortices, vortex dynamics, dynamic systems, asymptotic analysis, algebraic reduction.*

Introduction. It is thought that the most comprehensive study of the problem of motion of three point vortices with zero total circulation is presented in [1]. In this study, the initial six-dimensional Hamiltonian system describing vortex motion is reduced to a two-dimensional dynamic system using the well-known integrals of motion and elementary geometrical representations; equilibrium positions of this system are studied and an asymptotic analysis of vortex motion near the equilibrium positions is performed. Since the publication of this paper (1989), there has been little progress in studying three-vortex motion with zero total circulation. Some advances have been made in recent years in studies of the algebraic structure of the problem of three vortices and in the development of adequate methods for the analysis of the problem considered. Borisov [2] proposed an algebraic reduction method which uncovers the Lie algebraic structure of the equations of vortex motion and allows their study by Lie algebra methods. An important feature of this method is the canonicity of the reduced system, which makes it natural to use the variables angle–action in the problem.

In the present study, which is a continuation of the cycle of papers on the asymptotics of three-vortex motion [3–5], both the results of Rott [1] and a number of new results were obtained using the reduction scheme proposed in [2] and a modification of perturbation theory for Hamiltonian systems. The conclusion and representation of the results are clear and obvious, unlike in [1], where an inadequate reduction method is used.

1. Equations of Motion and the First Integrals. The motion of three point vortices is described by the Hamiltonian dynamic system

$$\dot{x}_i = \{x_i, H\}, \qquad \dot{y}_i = \{y_i, H\} \tag{1}$$

with the Hamiltonian

$$H = -\frac{1}{4\pi}\,(\gamma_1\gamma_2 \ln m_3 + \gamma_2\gamma_3 \ln m_1 + \gamma_3\gamma_1 \ln m_2) \tag{2}$$

and the Poisson bracket

$$\{F, G\} = \sum_{i=1}^{3} \frac{1}{\gamma_i}\Big(\frac{\partial F}{\partial x_i}\,\frac{\partial G}{\partial y_i} - \frac{\partial G}{\partial x_i}\,\frac{\partial F}{\partial y_i}\Big). \tag{3}$$

Here x_i and y_i are the cartesian coordinates of the ith vortex, γ_i is the circulation of the ith vortex, and $m_i = (x_j - x_k)^2 + (y_j - y_k)^2$ is the squared distance between vortices that are different from the ith vortex. In addition to Hamiltonian, system (1) has three independent first integrals

$$P = \sum_{i=1}^{3} \gamma_i x_i, \qquad Q = \sum_{i=1}^{3} \gamma_i y_i, \qquad I = \sum_{i=1}^{3} \gamma_i (x_i^2 + y_i^2), \tag{4}$$

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which are related to the invariance of the equations of motion under shifts and rotations of the coordinate system. Integrals (4) are not in involution, but they can be used to construct a pair of involutive integrals, for example,

$$P^2 + Q^2, \qquad D = \frac{(\gamma_1 + \gamma_2 + \gamma_3)I - P^2 - Q^2}{\gamma_1\gamma_2\gamma_3} = \frac{m_1}{\gamma_1} + \frac{m_2}{\gamma_2} + \frac{m_3}{\gamma_3}, \tag{5}$$

which allows one to reduce the order of the system by four units.

2. Reduction. System (1)–(3) describes the absolute motion of vortices. The relative motion of the vortices is associated with a reduced dynamic system. We denote by $m_0 = (x_1 - x_2)(y_2 - y_3) - (y_1 - y_2)(x_2 - x_3)$ the doubled oriented area of the vortex triangle and by $F = 4m_0^2 + m_1^2 + m_2^2 + m_3^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)$ the function implied by the Heron relation between the area of a triangle and the lengths of its sides.

The reduced system is defined as a constraint on the symplectic leaf $D = \text{const}$, $F = 0$ of a Hamiltonian system with coordinates m_0, m_1, m_2, and m_3, Hamiltonian (2), and the Poisson bracket

$$\{m_i, m_j\} = -4a_k m_0, \qquad \{m_0, m_i\} = (a_k - a_j)m_i + (a_j + a_k)(m_k - m_j),$$

where $a_i = \gamma_i^{-1}$ is the reverse circulation of the ith vortex; the set of indices (i, j, k) (if their range is not specified) takes all cyclic permutations of the triple of numbers $(1, 2, 3)$. This bracket is linear, and its corresponding Lie algebra is referred to as vortex algebra.

Using a linear transformation of vortex algebra, a symplectic leaf Is transformed to an ellipsoid ($a > 0$), paraboloid ($a = 0$) or hyperboloid ($a < 0$), depending on the sign of the quantity $a = a_1a_2 + a_2a_3 + a_3a_1$. The case of nonzero total circulation of the vortices considered in the present work corresponds to the value $a = 0$.

3. Canonical Coordinates. We introduce the new generators of the vortex algebra

$$e_0 = \frac{a_1m_1 + a_2m_2 + a_3m_3}{g}, \quad e_1 = \frac{m_0}{g}, \quad e_2 = \frac{m_3}{2g(a_1 + a_2)},$$

$$e_3 = -\frac{(a_1 + a_2)(m_1 - m_2) + (a_1 - a_2)m_3}{2g(a_1 + a_2)},$$

where $g \neq 0$ is an arbitrary constant. It is directly verified that

$$\{e_0, e_i\} = 0, \quad \{e_1, e_2\} = g^{-1}e_3, \quad \{e_2, e_3\} = g^{-1}e_1, \quad \{e_3, e_1\} = -g^{-1}e_0 \tag{6}$$

and, since $F = 0$,

$$e_1^2 + e_3^2 - 2e_0e_2 = 0.$$

Thus, in the coordinates e_1, e_2, and e_3, the phase space of the reduced system is a paraboloid. Transformation to the initial generators is carried out by the formulas

$$m_0 = ge_1, \qquad m_1 = g\frac{e_0 + 2a_2^2e_2 - 2a_2e_3}{a_1 + a_2}, \qquad m_2 = g\frac{e_0 + 2a_1^2e_2 + 2a_1e_3}{a_1 + a_2}; \tag{7}$$

$$m_3 = 2g(a_1 + a_2)e_2. \tag{8}$$

The projection $(e_1, e_2, e_3) \to (e_1, e_3)$ transforms the bracket (6) with accuracy up to the coefficient into the canonical bracket

$$\{e_3, e_1\} = -g^{-1}e_0. \tag{9}$$

Next, we shall set $g = D$ and call e_1 and e_3 normalized canonical coordinates.

A typical phase portrait of the reduced system in the normalized canonical coordinates is given in Fig. 1. The phase portrait has three singular points on the straight line $e_1 = 0$ and two hyperbolic points connected by the separatrix. The singular points on the straight line $e_1 = 0$ correspond to singular configurations of the vortices, the hyperbolic points to equilateral configurations, and the collinear configuration of vortices corresponds to a circle with an infinite radius, i.e., an infinitely distant point of the expanded phase plane. (A singular configuration of three vortices is a configuration in which two vortices are infinitely close to each other so that they can be considered merged. The collinear configuration of vortices is a configuration in which the vortices are on the same straight line.)

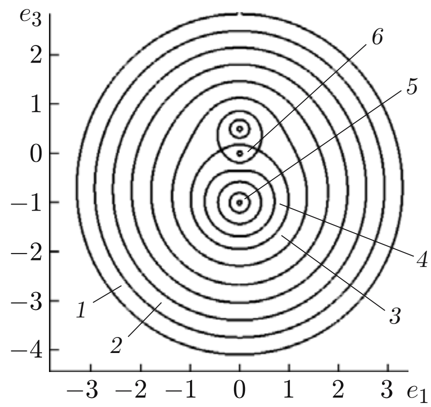

Fig. 1. Phase portrait of the reduced system in the normalized canonical coordinates for $\gamma_1 = 1$, $\gamma_2 = 0.5$, and $D = 1$: points 1–6 correspond to the vortex trajectories shown in Fig. 2a–f, respectively.

4. Relationship between the Absolute and Relative Coordinates. We set $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$. The following relations hold:

$$x_i P + y_i Q = (I - \gamma_k m_j - \gamma_j m_k)/2, \qquad y_{ij} P - x_{ij} Q = \gamma_k m_0. \tag{10}$$

For $P = 0$ and $Q \neq 0$, from relations (10) we find

$$x_{ij} = -\gamma_k m_0 / Q, \qquad y_i = (I - \gamma_k m_j - \gamma_j m_k)/(2Q). \tag{11}$$

Thus, one quadrature is required to retrieve the absolute coordinates. This quadratures can be taken, for example, to be the equation for x_3 from (1) with the substitution of y_{23} and y_{31} from relations (11).

5. Numerical Modeling. Figure 2 shows vortex trajectories obtained by numerical solution of the dynamics equations (1)–(3) and corresponding to the points of the phase portrait of the reduced system shown in Fig. 1. All these trajectories are attached to a coordinate system in which the vortices describe closed curves [2]. Figure 2 provides a complete idea of the trajectories of the vortices for any of their initial position.

6. Motion in the Collinear Configuration. For $P = Q = 0$, the absolute dynamics is retrieved from the relative dynamics without quadratures. This case corresponds to the motion of vortices in the collinear configuration.

From relations (4) and (10), we obtain

$$\gamma_i x_{ij} = \gamma_k x_{jk}, \qquad \gamma_i y_{ij} = \gamma_k y_{jk}; \tag{12}$$

$$m_0 = 0, \qquad m_i = -\gamma_i^2 I/(\gamma_1\gamma_2\gamma_3). \tag{13}$$

Using relations (12) and (13), we write the equations of vortex dynamics (1)–(3) as

$$\dot{x}_k = -\frac{\gamma_{ij}\gamma_k}{\gamma}\,\Omega y_{ij}, \qquad \dot{y}_k = \frac{\gamma_{ij}\gamma_k}{\gamma}\,\Omega x_{ij}, \qquad \Omega = -\frac{\gamma}{2\pi I} \tag{14}$$

or

$$\dot{x}_{ij} = \Omega y_{ij}, \qquad \dot{y}_{ij} = -\Omega x_{ij}, \tag{15}$$

where $\gamma_{ij} = \gamma_i - \gamma_j$ and $\gamma = \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1$. Assuming that, at $t = 0$, the vortices are located on the y axis, from relations (15) we obtain

$$x_{ij} = A_{ij}\sin\Omega t, \qquad y_{ij} = A_{ij}\cos\Omega t. \tag{16}$$

By virtue of (13), we set

$$A_{12} = \gamma_3\sqrt{-\frac{I}{\gamma_1\gamma_2\gamma_3}}. \tag{17}$$

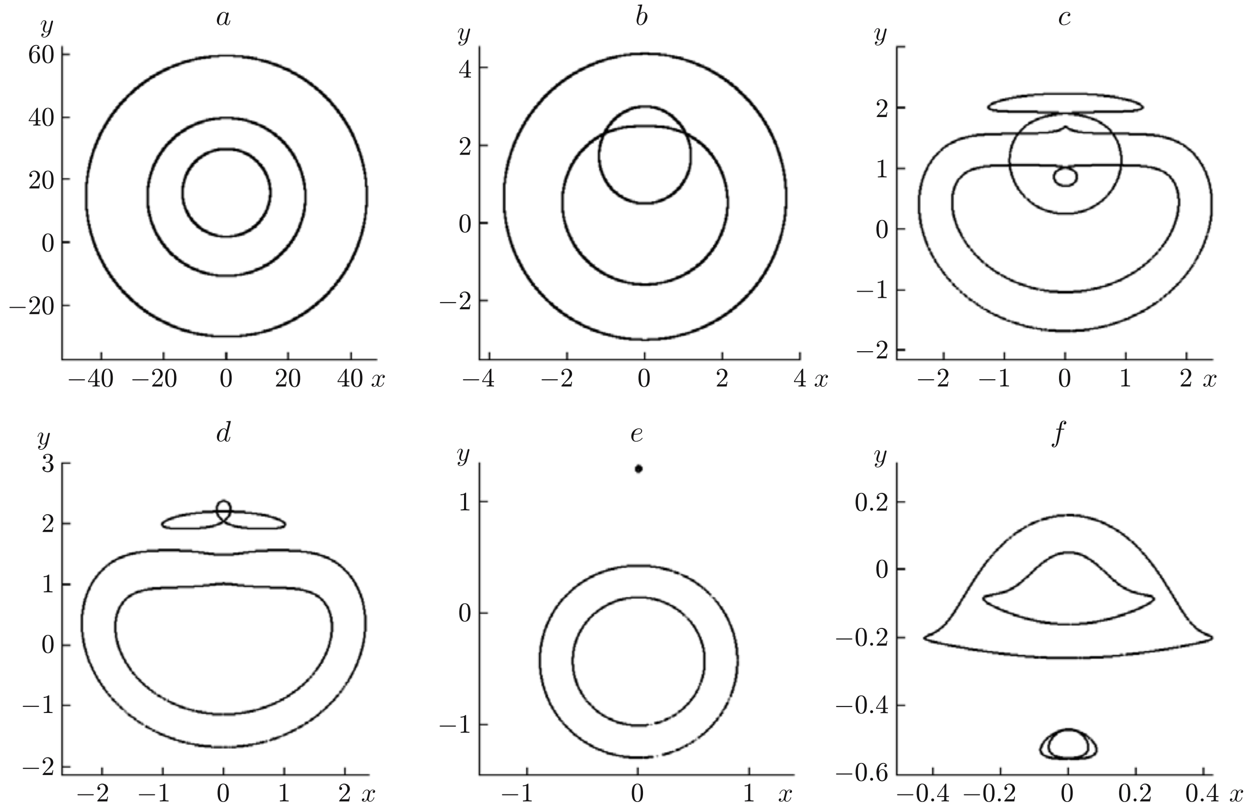

Fig. 2. Vortex trajectories in moving Cartesian coordinates: nearly collinear configurations (a and b), nearly equilateral configurations (c, d, and f), and nearly singular configurations (e).

Then, from expressions (12) and (16), it follows that the remaining values of A_{ij} are obtained from (17) by cyclic permutation of the indices. Integrating (14), we finally obtain

$$x_k = r_k \sin \Omega t, \qquad y_k = r_k \cos \Omega t + y_{k0},$$

where

$$r_k = -\frac{\gamma_{ij}\gamma_k^2}{\gamma}\sqrt{-\frac{I}{\gamma_1\gamma_2\gamma_3}}$$

and y_{k0} are integration constants which satisfy relations (12).

Thus, the vortices in the collinear configuration rotate at angular velocity Ω around the common center located on the straight line connecting these vortices.

7. Relative Motion in a Vicinity of a Singular Configuration. In nearly singular and collinear configurations, the vortex dynamics is determined as follows. First, the reduced dynamics is determined by calculating the asymptotics of the variables angle–action in the vicinity of the corresponding singular point (zero or infinity). Then, using the relationship between the absolute and relative coordinates, the asymptotics of the absolute motion is determined.

Let us find the asymptotics of the variables angle–action (Θ, J) in the vicinity of the singular points. Without loss of generality, we confine the consideration to the singular point located at the center of the system of the normalized canonical coordinates. We set

$$e_1 = \rho \cos \varphi, \qquad e_3 = \rho \sin \varphi \tag{18}$$

and assume that a trajectory close to the singular point is given by the expansion

$$\rho = \varepsilon + \rho_2\varepsilon^2 + \rho_3\varepsilon^3 + \rho_4\varepsilon^4 + O(\varepsilon^5), \tag{19}$$

where ε is an expansion parameter and ρ_i are functions of φ. We also assume that the following expansion is valid:

$$h = h_\infty \ln \varepsilon + h_0 + h_1 \varepsilon + \ldots + O(\varepsilon^5), \tag{20}$$

where h_∞ and h_i are constants. The expansion coefficients are determined from the conditions

$$H(\rho, \varphi) = h; \tag{21}$$

$$J \equiv \frac{g}{4\pi} \int\limits_0^{2\pi} \rho^2 \, d\varphi = \frac{1}{2} g \varepsilon^2 + O(\varepsilon^6). \tag{22}$$

Substituting (19) and (20) into relations (21) and (22), separating the coefficients at identical powers of ε, and solving the equations obtained, we find

$$h_\infty = -\frac{\gamma_1\gamma_2}{2\pi}, \qquad h_1 = h_2 = h_3 = 0, \qquad h_4 = \frac{3\gamma_3^4}{16\pi\gamma_2^3\gamma_1^3},$$

$$\rho_2 = 0, \qquad \rho_3 = \frac{\gamma_3^2 \cos 2\varphi}{2\gamma_1^2\gamma_2^2}, \qquad \rho_4 = \frac{\gamma_{12}\gamma_3^2 \sin 3\varphi}{3\gamma_1^3\gamma_2^3}.$$

The generating function of the transformation to the variables angle–action and the angular variable are given by the relations

$$S = \frac{g}{2} \int\limits_{\pi/2}^{\varphi} \rho^2 \, d\varphi, \qquad \theta = -\frac{\partial S}{\partial J} = -\frac{\partial S}{\partial \varepsilon} \Big/ \frac{\partial J}{\partial \varepsilon},$$

in which the value of $\pi/2$ for the lower limit of integration is chosen for convenience. Omitting the expression for S, we have

$$\theta = \frac{\pi}{2} - \varphi - \frac{\gamma_3^2 \sin 2\varphi}{\gamma_1^2\gamma_2^2} \varepsilon^2 + \frac{5\gamma_{12}\gamma_3^2 \cos 3\varphi}{9\gamma_1^3\gamma_2^3} \varepsilon^3 + O(\varepsilon^4).$$

Finally, the relationship between the canonical coordinates and the coordinates angle–action is given by the relations

$$\varphi = \frac{\pi}{2} - \theta - \frac{\gamma_3^2 \sin 2\theta}{\gamma_1^2\gamma_2^2} \varepsilon^2 - \frac{5\gamma_{12}\gamma_3^2 \sin 3\theta}{9\gamma_1^3\gamma_2^3} \varepsilon^3 + O(\varepsilon^4),$$

$$\rho = \varepsilon + \frac{\gamma_3^2 \cos 2\theta}{2\gamma_1^2\gamma_2^2} \varepsilon^3 + \frac{\gamma_{12}\gamma_3^2 \sin 3\theta}{3\gamma_1^3\gamma_2^3} \varepsilon^4 + O(\varepsilon^5)$$

together with (18) and (22). Using the expression for the Hamiltonian in the variables angle–action, we obtain the frequency of the relative motion of three vortices:

$$\omega \equiv \frac{\partial h}{\partial J} = -\frac{\gamma_1\gamma_2}{2\pi g} \varepsilon^{-2} + O(\varepsilon^2).$$

8. Absolute Motion in the Vicinity of the Singular Configuration. Under the assumption $P = 0$ for the Cartesian coordinates of the first and third vortices normalized on g/Q, we have the expressions

$$\bar{x}_1 = \bar{v}t + \gamma_2 \sin\theta\, \varepsilon + \frac{\gamma_3^2(\sin 3\theta + 3\sin\theta)}{4\gamma_1^2\gamma_2} \varepsilon^3 + O(\varepsilon^4); \tag{23}$$

$$\bar{y}_1 = \bar{y}_0 + \frac{\gamma_1\gamma_2}{2} + \gamma_2 \cos\theta\, \varepsilon + \frac{\gamma_3^2(\cos 3\theta - 3\cos\theta)}{4\gamma_1^2\gamma_2} \varepsilon^3 + O(\varepsilon^4); \tag{24}$$

$$\bar{x}_3 = \bar{v}t - \frac{\gamma_3^2 \sin 2\theta}{2\gamma_1^2\gamma_2^2} \varepsilon^4 - \frac{\gamma_{12}\gamma_3^2 \sin 3\theta}{3\gamma_1^3\gamma_2^3} \varepsilon^5 + O(\varepsilon^6); \tag{25}$$

$$\bar{y}_3 = \bar{y}_0 - \frac{\gamma_1\gamma_2}{2} + \frac{\gamma_3^2 \cos 2\theta}{2\gamma_1^2\gamma_2^2} \varepsilon^4 + \frac{\gamma_{12}\gamma_3^2 \cos 3\theta}{3\gamma_1^3\gamma_2^3} \varepsilon^5 + O(\varepsilon^6). \tag{26}$$

Here

$$\bar{v} = \frac{\gamma_3^2}{2\pi g} - \frac{3\gamma_3^4}{4\pi g\gamma_1^3\gamma_2^3}\,\varepsilon^4 + O(\varepsilon^6), \qquad \bar{y}_0 = \frac{1}{2}\,g^{-1}I - \frac{1}{2}\,\varepsilon^2;$$

the bar denotes normalized quantities. The expressions for the coordinates of the second vortex are similar to expressions (23) and (24) with the cyclic replacement of the indices 1 and 2 and rotation θ by the angle π. In this case, the integration constant, which is common for all horizontal coordinates, is set equal to zero.

Analyzing the initial approximations of the coordinates, it can be concluded that in a coordinate system moving along the x axis at velocity v, the first two vortices rotate at angular velocity ω on circles around their circulation center and the third vortex rotates at velocity 2ω around the point located at the distance $\bar{r} = \gamma_1\gamma_2$ from the first two vortices. In view of (5), for the angular velocity ω we obtain the relation

$$\omega = \frac{\gamma_1 + \gamma_2}{2\pi(r_1 + r_2)^2},$$

which coincides with the expression for the angular velocity of rotation of two free vortices. This indicates that the third vortex does not influence the first two vortices.

To analyze the subsequent approximations for each vortex, we transform to the coordinate system in which the vortex is at rest in the initial approximation. In a coordinate system rotating at velocity ω around the circulation center of the first two vortices, with accuracy up to ε^4, we have

$$\bar{x}_1 = \frac{\gamma_3^2\sin 2\theta}{\gamma_1^2\gamma_2}\,\varepsilon^3, \qquad \bar{y}_1 = \gamma_2\varepsilon - \frac{\gamma_3^2\cos 2\theta}{2\gamma_1^2\gamma_2}\,\varepsilon^3,$$

$$\bar{x}_2 = -\frac{\gamma_3^2\sin 2\theta}{\gamma_1\gamma_2^2}\,\varepsilon^3, \qquad \bar{y}_2 = -\gamma_1\varepsilon + \frac{\gamma_3^2\cos 2\theta}{2\gamma_1\gamma_2^2}\,\varepsilon^3,$$

i.e., the vortices rotate at velocity 2ω on ellipses with eccentricity 1/2. In transformation to a coordinate system rotating at velocity 2ω around the point with coordinates

$$\bar{x} = 0, \qquad \bar{y} = -\gamma_1\gamma_2/2 + \bar{A}\varepsilon^5$$

(A is an arbitrary constant), for the third vortex, with accuracy up to ε^6, we obtain

$$\bar{x}_3 = -\frac{3\gamma_1^3\gamma_2^3\bar{A}\sin 2\theta + \gamma_{12}\gamma_3^2\sin\theta}{3\gamma_1^3\gamma_2^3}\,\varepsilon^5,$$

$$\bar{y}_3 = \frac{\gamma_3^2}{2\gamma_1^2\gamma_2^2}\,\varepsilon^4 - \frac{3\gamma_1^3\gamma_2^3\bar{A}\cos 2\theta - \gamma_{12}\gamma_3^2\cos\theta}{3\gamma_1^3\gamma_2^3}\,\varepsilon^5.$$

For the values of $|A|$ varied from zero to fairly large values, the limiting trajectories correspond to circles, and the entire set of intermediate trajectories includes the trajectories given in Fig. 3a and b. The trajectories in Fig. 3a and b correspond to values $A = A_1$; $A = A_1/2$, where $\bar{A}_1 = \gamma_{12}\gamma_3^2/(6\gamma_1^3\gamma_2^3)$.

For $\gamma_1 = \gamma_2$, in expansions (25) and (26), terms of the order of ε^5 vanish. In this case, in a coordinate system rotating at velocity 2ω around the point with coordinates

$$\bar{x} = 0, \qquad \bar{y} = -\gamma_1\gamma_2/2 + \bar{A}\varepsilon^6,$$

we have

$$x_3 = -\frac{(\gamma_1^4\bar{A} + 11)\sin 2\theta}{\gamma_1^4}\,\varepsilon^6 - \frac{16\sin 4\theta}{\gamma_1^6}\,\varepsilon^8,$$
$$y_3 = \frac{2}{\gamma_1^2}\,\varepsilon^4 - \frac{(\gamma_1^4\bar{A} + 5)\cos 2\theta}{\gamma_1^4}\,\varepsilon^6 - \frac{2(\cos 4\theta - 12)}{\gamma_1^6}\,\varepsilon^8. \qquad (27)$$

Thus, the vortex trajectory is an ellipse. For $\bar{A} = -5/\gamma_1^4$, the second term in expression (27) vanishes; the corresponding trajectory is given in Fig. 3c.

9. Relative Motion in the Vicinity of the Collinear Configuration. The algorithm for calculating the variables angle–action on the periphery of the phase plane coincides with the corresponding similar algorithm in the vicinity of the singular point. The data and results differ.

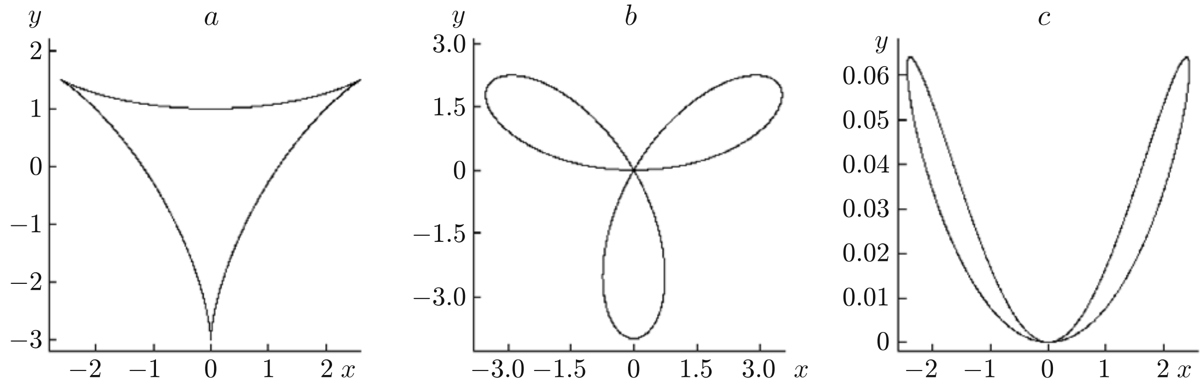

Fig. 3. Trajectory of motion of the third vortex in a special coordinate system in Cartesian coordinates: (a, b) $\gamma_1 = 1$, $\gamma_2 = 0.5$, $D = 1$, canonical coordinates $x_{10} = x_{20} = 0$, $y_{10} = -y_{20} = 0.003$, and $A = A_1$ (a), and $A_1/2$ (b); (c) $\gamma_1 = \gamma_2 = 1$, $D = 1$, $x_{10} = x_{20} = 0$, $y_{10} = -y_{20} = 0.01$, and $\bar{A} = -5/\gamma_1^4$.

Let us assume that in polar coordinates, the expression for the phase curves become

$$\rho = 1/\varepsilon + \rho_0 + \rho_1\varepsilon + \rho_2\varepsilon^2 + \rho_3\varepsilon^3 + O(\varepsilon^4), \tag{28}$$

the Hamiltonian admits the expansion

$$h = h_\infty \ln\varepsilon + h_0 + h_2\varepsilon^2 + h_4\varepsilon^4 + O(\varepsilon^6), \tag{29}$$

and the action is related to the small parameter by the relation

$$J = \frac{g}{2\varepsilon^2} + O(\varepsilon^3). \tag{30}$$

Using formulas (28)–(30), we obtain

$$\rho_0 = \frac{\gamma_{12}\gamma_3^2\sin\varphi}{\gamma}, \qquad \rho_1 = -\frac{(\gamma_1^4 + 4\gamma_1^2\gamma_2^2 + \gamma_2^4)\gamma_3^2\cos 2\varphi + \gamma_{12}^2\gamma_3^4}{4\gamma^2}; \tag{31}$$

$$\rho_2 = -\frac{\gamma_1^2\gamma_2^2\gamma_3^2\gamma_{12}(\gamma_{12}^2\sin 3\varphi + 9\gamma_3^2\sin\varphi)}{6\gamma^3}; \tag{32}$$

$$h_\infty = \frac{\gamma}{2\pi}, \qquad h_2 = 0, \qquad h_4 = -\frac{9\gamma_1^4\gamma_2^4\gamma_3^4}{16\pi\gamma^3}; \tag{33}$$

$$\varphi = \frac{1}{2}\pi - \theta + \frac{\gamma_{12}\gamma_3^2\sin\theta}{\gamma}\,\varepsilon - \frac{\gamma_{12}^2\gamma_3^4\sin 2\theta}{2\gamma^2}\,\varepsilon^2$$

$$-\frac{\gamma_{12}\gamma_3^2}{2\gamma^3}\Big[\gamma_{23}\gamma_{31}\Big(\frac{1}{3}\,\gamma^2 - \frac{5}{6}\,\gamma_1\gamma_2\gamma_3^2 + \frac{1}{9}\,\gamma_1^2\gamma_2^2\Big)\sin 3\theta - \frac{3}{2}\,\gamma_1^2\gamma_2^2\gamma_3^2\sin\theta\Big]\varepsilon^3 + O(\varepsilon^4). \tag{34}$$

The expression for r_3 is not given here because of space limitations.

Differentiating the Hamiltonian (29), (33) with respect to J, we find the frequency of the relative motion

$$\omega = -\frac{\gamma}{2\pi g}\,\varepsilon^2 + O(\varepsilon^6). \tag{35}$$

10. Absolute Motion in the Vicinity of the Collinear Configuration. Repeating the calculations of the absolute coordinates of the vortices in a nearly singular configuration using expressions (28), (30)–(32), and (34) for the relative coordinates, with accuracy up to ε^3 we obtain

$$\bar{y}_3 + i\bar{x}_3 = \bar{y}_{30} - \frac{\gamma_{12}\gamma_3^2 \, \mathrm{e}^{i\omega t}}{\gamma\varepsilon} - \frac{3\gamma_1^2\gamma_2^2\gamma_3^2 \, \mathrm{e}^{i2\omega t}}{2\gamma^2} - \frac{\gamma_{12}\gamma_1^2\gamma_2^2\gamma_3^2((7\gamma - 15\gamma_1\gamma_2)\, \mathrm{e}^{3i\omega t} + 9\gamma_3^2 \, \mathrm{e}^{-i\omega t})}{12\gamma^3}\, \varepsilon$$

$$+ \frac{\gamma_1^2\gamma_2^2\gamma_3^2}{\gamma^4} \Big[\Big(\frac{13}{36}\gamma^3 - \frac{4}{9}\gamma_1\gamma_2\gamma^2 - \frac{61}{24}\gamma_1^2\gamma_2^2\gamma + \frac{93}{24}\gamma_1^3\gamma_2^3\Big)\, \mathrm{e}^{4i\omega t} - \frac{2}{9}\gamma_3^2\gamma_{12}^2\gamma_{31}\gamma_{23}\, \mathrm{e}^{-2i\omega t}\Big]\, \varepsilon^2,$$

where

$$\bar{y}_{30} = \frac{1}{2}\, g^{-1} I - \frac{1}{2}\, \varepsilon^{-2} + \frac{\gamma^3 - 5\gamma_1^2\gamma_2^2\gamma + 3\gamma_1^3\gamma_2^3}{2\gamma^2}.$$

The coordinates of the remaining vortices are obtained by cyclic permutation of the indices.

In the approximations considered, the horizontal precession of the vortices is absent. Using calculations performed with higher accuracy, for the precession velocity we obtain

$$v = \frac{\gamma_1^4\gamma_2^4\gamma_3^4}{\pi\gamma^3 Q}\, \varepsilon^4 + O(\varepsilon^5).$$

From the obtained expressions for the coordinates, it follows that in the initial approximation, the vortices forming the collinear configuration rotate at the angular velocity (35) around the point

$$\bar{x} = 0, \qquad \bar{y} = g^{-1} I/2 - \varepsilon^{-2}/2 \tag{36}$$

at a distance from this point equal to

$$\bar{r}_k = -\gamma_{ij}\gamma_k^2/(\gamma\varepsilon).$$

If, in (36), we set $y \sim 0$, which is equivalent to the condition

$$\varepsilon \sim \sqrt{g/I}, \tag{37}$$

the obtained vortex dynamics will coincide with that considered above for $P = Q = 0$. Asymptotics (37) gives an idea of the relation between the integrals D and I at which a nearly collinear configuration occurs. Such a configuration occurs for any values of the momentum of the vortices and not only for small values, as in [1].

To describe the motion in the next approximation, we transform to a coordinate system rotating at velocity ω in which the vortices in the initial approximation are at rest and the center of rotation can be any point with the coordinates

$$x = 0, \qquad y = A + \varepsilon B + \varepsilon^2 C$$

(A, B, and C are arbitrary constants). In this coordinate system, with accuracy up to ε, we have

$$x_k = A_k \sin\omega t, \qquad \bar{y}_k = -\gamma_{ij}\gamma_k^2/(\gamma\varepsilon) + \bar{B}_k \cos\omega t,$$

where

$$\bar{B}_3 = -\bar{A} + (\gamma_1^4 + 2\gamma_1^3\gamma_2 + \gamma_1^2\gamma_2^2 + 2\gamma_1\gamma_2^3 + \gamma_2^4)/(2\gamma),$$

and the remaining coefficients are found from the relations

$$\bar{A}_k + \bar{B}_k = -3\gamma_1^2\gamma_2^2\gamma_3^2/\gamma^2, \qquad \bar{A}_i - \bar{A}_j = -\gamma_{ij}\gamma_k^3/\gamma, \tag{38}$$

which do not depend on the displacement A which defines the coordinate system.

Thus, in any of the specified coordinate systems, the vortices rotate at velocity ω on ellipses. The rotation directions depend on the circulation, integrals, and displacement A and can be arbitrary. According to (39), an invariant under the vortex displacement and number is the quantity $\alpha + (-1)^s\beta$, where α and β are the major and minor axes of the ellipse and $s = 0$ or 1 (according to the direction of vortex rotation).

Figure 4 shows vortex trajectories in a rotating coordinate system for various displacements A normalized to the same center.

If the length of the vertical axis of one of the ellipses is equal to zero, a description of vortex motion requires additional terms of the asymptotics. For $B_3 = 0$, we have

$$\bar{x}_3 = -\frac{3\gamma_1^2\gamma_2^2\gamma_3^2 \sin\omega t}{\gamma^2} + O(\varepsilon),$$

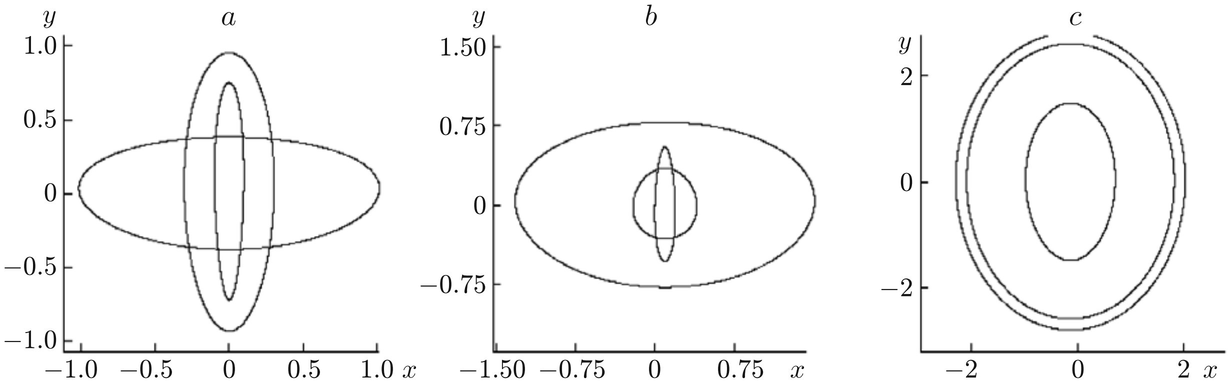

Fig. 4. Trajectories of vortices in special coordinate systems in Cartesian coordinates for $\gamma_1 = 1$, $\gamma_2 = 0.5$, $D = 1$, $x_{10} = x_{20} = 0$, $y_{10} = -y_{20} = 20$, and various displacements $A = -5$ (a), 10 (b), and 100 (c).

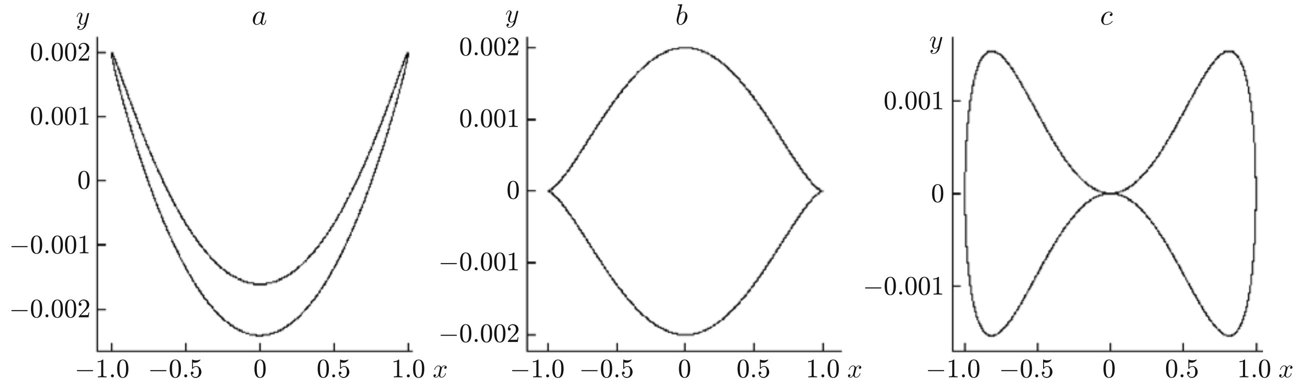

Fig. 5. Trajectory of motion of the third vortex in special coordinate system in Cartesian coordinates: (a) $\gamma_1 = 1$, $\gamma_2 = 0.25$, and $D = 10$; (b) $\gamma_1 = 1$, $\gamma_2 = 0.85$, and $D = 1$; (c) $\gamma_1 = 1$, $\gamma_2 = 2.5$, and $D = 1$.

$$\bar{y}_3 = -\frac{\gamma_{ij}\gamma_k^2}{\gamma\varepsilon} - \frac{\gamma_1^2\gamma_2^2\gamma_3^2\gamma_{12}^3\cos 2\omega t + 6\bar{B}\gamma^3\cos\omega t}{6\gamma^3}\,\varepsilon$$

$$-\frac{\gamma_1^2\gamma_2^2\gamma_3^2(\gamma^3/12 + 2\gamma_1\gamma_2\gamma^2/3 - \gamma_1^2\gamma_2^2\gamma/8 - 15\gamma_1^3\gamma_2^3/8)\cos 3\omega t + \bar{C}\gamma^4\cos\omega t}{\gamma^4}\,\varepsilon^2 + O(\varepsilon^3).$$

If the circulations γ_1 and γ_2 differ significantly, one can omit the term containing ε^2 in the last expression. In this case, the vortex trajectory varies from a parabola for $B = 0$ to an ellipse for large $|B|$. An intermediate trajectory is given in Fig. 5a. If the circulations γ_1 and γ_2 are close and $B = 0$, the term with ε can be omitted. In this case, as C is varied, the vortex trajectory changes from ellipse to ellipse, including the trajectories shown in Fig. 5b and c. The trajectories shown in Fig. 5b and c correspond to the values of C for which the ratios of the amplitudes of the third and first harmonics in the expression for y_3 are equal to $1/1$ and $-1/3$, respectively. In Cartesian coordinates with appropriate normalization, these trajectories are given by the equations

$$y^2 = (1 - x^2)^3, \qquad y^2 = x^4(1 - x^2).$$

It should be noted that equations of this type also arise in a description of motion of three vortices if the trajectory of one of them passes through the circulation center [5].

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